

MVW-EXTENSIONS OF REAL QUATERNIONIC CLASSICAL GROUPS

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ABSTRACT. Let G be a real quaternionic classical group $\mathrm{GL}_n(\mathbb{H})$, $\mathrm{Sp}(p, q)$ or $\mathrm{O}^*(2n)$. We define an extension \check{G} of G with the following property: it contains G as a subgroup of index two, and for every $x \in G$, there is an element $\check{g} \in \check{G} \setminus G$ such that $\check{g}x\check{g}^{-1} = x^{-1}$. This is similar to Moeglin-Vigneras-Waldspurger's extensions of non-quaternionic classical groups.

1. INTRODUCTION

We follow the terminology of [SZ]. So by a commutative involutive algebra, we mean a commutative semisimple finite dimensional \mathbb{R} -algebra, equipped with a \mathbb{R} -algebra involution on it. A commutative involutive algebra is said to be simple if it is nonzero, and has no involution stable ideal except for $\{0\}$ and itself. Every commutative involutive algebra is uniquely a product of simple ones, and every simple one is isomorphic to one of the followings:

$$(1) \quad (\mathbb{R}, 1_{\mathbb{R}}), (\mathbb{C}, 1_{\mathbb{C}}), (\mathbb{C}, \bar{}_{\mathbb{C}}), (\mathbb{R} \times \mathbb{R}, \tau_{\mathbb{R}}), (\mathbb{C} \times \mathbb{C}, \tau_{\mathbb{C}}),$$

where $1_{\mathbb{R}}$ and $1_{\mathbb{C}}$ are the identity maps, and $\tau_{\mathbb{R}}$ and $\tau_{\mathbb{C}}$ are the maps which interchange the coordinates. In this paper, we use “ $\bar{}$ ” (or $\bar{}_{\mathbb{C}}$ and $\bar{}_{\mathbb{H}}$) to indicate complex conjugations and quaternion conjugations.

Let $\epsilon = \pm 1$. Let (A, τ) be a commutative involutive algebra. Let E be a finitely generated A -module. Recall that an (ϵ, τ) -hermitian form on E is a non-degenerate \mathbb{R} -bilinear map

$$\langle , \rangle_E : E \times E \rightarrow A$$

satisfying

$$(2) \quad \langle au, v \rangle_E = a \langle u, v \rangle_E, \quad \langle u, v \rangle_E = \epsilon \langle v, u \rangle_E^{\tau}, \quad a \in A, u, v \in E.$$

Equipped with such a form, we call E an (ϵ, τ) -hermitian A -module. Then we denote by $\mathrm{U}(E)$ the group of A -module automorphisms of E preserving the form \langle , \rangle_E. If (A, τ) is simple, then E is free as an A -module, and $\mathrm{U}(E)$ is a non-quaternionic classical group as in the following table.

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TABLE 1.

(A, τ)	$(\mathbb{R}, 1_{\mathbb{R}})$	$(\mathbb{C}, 1_{\mathbb{C}})$	$(\mathbb{C}, \bar{\cdot}_{\mathbb{C}})$	$(\mathbb{R} \times \mathbb{R}, \tau_{\mathbb{R}})$	$(\mathbb{C} \times \mathbb{C}, \tau_{\mathbb{C}})$
$\epsilon = 1$	$O(p, q)$	$O_n(\mathbb{C})$	$U(p, q)$	$GL_n(\mathbb{R})$	$GL_n(\mathbb{C})$
$\epsilon = -1$	$Sp_{2n}(\mathbb{R})$	$Sp_{2n}(\mathbb{C})$	$U(p, q)$	$GL_n(\mathbb{R})$	$GL_n(\mathbb{C})$

As in [SZ], we define the MVW-extension $\check{U}(E)$ (named after Moeglin, Vigneras and Waldspurger) of $U(E)$ to be the subgroup of $GL(E_{\mathbb{R}}) \times \{\pm 1\}$ consisting of pairs (g, δ) such that either

$$\delta = 1 \quad \text{and} \quad \langle gu, gv \rangle_E = \langle u, v \rangle_E, \quad u, v \in E,$$

or

$$\delta = -1 \quad \text{and} \quad \langle gu, gv \rangle_E = \langle v, u \rangle_E, \quad u, v \in E.$$

Here $E_{\mathbb{R}} := E$, viewed as a \mathbb{R} -vector space. Every $g \in GL(E_{\mathbb{R}})$ is automatically A -linear if $(g, 1) \in \check{U}(E)$, and is conjugate A -linear (with respect to τ) if $(g, -1) \in \check{U}(E)$.

The MVW-extension $\check{U}(E)$ has the following remarkable property ([MVW, Proposition 4.I.2]): it contains $U(E)$ as a subgroup of index two, and for every $x \in U(E)$, there is an element $\check{g} \in \check{U}(E) \setminus U(E)$ such that $\check{g}x\check{g}^{-1} = x^{-1}$. By character theory and Harish-Chandra's regularity theorem, this has the following interesting consequence in representation theory: for every element $\check{g} \in \check{U}(E) \setminus U(E)$, and for every irreducible Casselman-Wallach representation π of $U(E)$, its twist $\pi^{\check{g}}$ is isomorphic to its contragredient π^{\vee} . Here the twist $\pi^{\check{g}}$ is the representation $g \mapsto \pi(\check{g}g\check{g}^{-1})$ of $U(E)$ on the same space as that of π . The reader is referred to [Cas] and [Wal, Chapter 11] for details on Casselman-Wallach representations.

This paper is aimed to define similar extensions for real quaternionic classical groups. Denote by \mathbb{H} a fixed real quaternion algebra. Up to isomorphism, this is the unique central simple division algebra over \mathbb{R} of dimension 4. Now let E be a finitely generated A - \mathbb{H} -bimodule. Every A - \mathbb{H} -bimodule is assumed to satisfy that

$$tu = ut, \quad t \in \mathbb{R}, u \in E.$$

Here the first “ t ” is viewed as an element of A , and the second one is viewed as an element of \mathbb{H} . An (ϵ, τ) -hermitian form $\langle \cdot, \cdot \rangle_E$ on E is said to be quaternionic if it further satisfies that

$$(3) \quad \langle uh, v \rangle_E = \langle u, v\bar{h} \rangle_E, \quad h \in \mathbb{H}, u, v \in E.$$

We equip on E with such a form and then call E a quaternionic (ϵ, τ) -hermitian A -module.

Quaternion (ϵ, τ) -hermitian A -modules are classified as follows.

Proposition 1.1. *Assume that (A, τ) is simple. Then every quaternionic (ϵ, τ) -hermitian A -module is isomorphic to exactly one in the following table. Here p, q, n are nonnegative integers, and the quaternionic (ϵ, τ) -hermitian forms on the spaces in the table are given explicitly in Section 2.*

TABLE 2.

(A, τ)	$(\mathbb{R}, 1_{\mathbb{R}})$	$(\mathbb{C}, 1_{\mathbb{C}})$	$(\mathbb{C}, \bar{\cdot}_{\mathbb{C}})$	$(\mathbb{R} \times \mathbb{R}, \tau_{\mathbb{R}})$	$(\mathbb{C} \times \mathbb{C}, \tau_{\mathbb{C}})$
$\epsilon = 1$	\mathbb{H}^{p+q}	$\mathbb{C}^{2n} \otimes_{\mathbb{C}} \mathbb{H}$	$\mathbb{C}^{p+q} \otimes_{\mathbb{C}} \mathbb{H}$	$\mathbb{H}^n \oplus \mathbb{H}^n$	$\mathbb{H}^n \oplus \mathbb{H}^n$
$\epsilon = -1$	\mathbb{H}^n	$\mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{H}$	$\mathbb{C}^{p+q} \otimes_{\mathbb{C}} \mathbb{H}$	$\mathbb{H}^n \oplus \mathbb{H}^n$	$\mathbb{H}^n \oplus \mathbb{H}^n$

Denote by $U(E)$ the group of A - \mathbb{H} -bimodule automorphisms of E preserving the form $\langle \cdot, \cdot \rangle_E$. If (A, τ) is simple and E is as in Table 2, then $U(E)$ is respectively as in the following table.

TABLE 3.

(A, τ)	$(\mathbb{R}, 1_{\mathbb{R}})$	$(\mathbb{C}, 1_{\mathbb{C}})$	$(\mathbb{C}, \bar{\cdot}_{\mathbb{C}})$	$(\mathbb{R} \times \mathbb{R}, \tau_{\mathbb{R}})$	$(\mathbb{C} \times \mathbb{C}, \tau_{\mathbb{C}})$
$\epsilon = 1$	$\mathrm{Sp}(p, q)$	$\mathrm{Sp}_{2n}(\mathbb{C})$	$\mathrm{U}(p, q)$	$\mathrm{GL}_n(\mathbb{H})$	$\mathrm{GL}_n(\mathbb{C})$
$\epsilon = -1$	$\mathrm{O}^*(2n)$	$\mathrm{O}_n(\mathbb{C})$	$\mathrm{U}(p, q)$	$\mathrm{GL}_n(\mathbb{H})$	$\mathrm{GL}_n(\mathbb{C})$

In general, we define the MVW-extension of $U(E)$ to be

$$\check{U}(E) := \check{U}(E_A) \cap (\mathrm{GL}(E_{\mathbb{H}}) \times \{\pm 1\}),$$

where E_A is the underlying (ϵ, τ) -hermitian A -module of E (forgetting the \mathbb{H} -module structure), and $E_{\mathbb{H}} := E$, viewed as a right \mathbb{H} -vector space.

The main result of this paper is

Theorem 1.2. *Let E be a quaternionic (ϵ, τ) -hermitian A -module. Then $\check{U}(E)$ contains $U(E)$ as a subgroup of index two, and for every $x \in U(E)$, there is an element $\check{y} \in \check{U}(E) \setminus U(E)$ such that $\check{y}x\check{y}^{-1} = x^{-1}$.*

Remarks: (a) MVW-extensions are defined for p-adic non-quaternionic classical groups as well (cf. [Sun2]). One may also define MVW-extensions for p-adic quaternionic general linear groups (cf. [Rag, Lemma 3.1]). But by a private communication with D. Prasad, it seems that it is not possible to define MVW-extensions with desired properties for p-adic quaternionic unitary groups.

(b) MVW-extensions have many applications to representation theory of classical groups. For examples, they are used to prove Multiplicity One Theorems for non-quaternionic classical groups ([AGRS, SZ]), and multiplicity preservation for local theta correspondences ([LST]). They are also used to prove that local

theta correspondence maps hermitian representations to hermitian representations (cf. [Prz, Sun1]).

(c) The Lie algebra analog of the second statement of Theorem 1.2 also holds. Namely, for every x in the Lie algebra of $U(E)$, there is an element $\check{g} \in \check{U}(E) \setminus U(E)$ such that $\text{Ad}_{\check{g}}x = -x$.

(d) If $\epsilon = 1$ and $(A, \tau) = (\mathbb{R}, 1_{\mathbb{R}})$ or $(\mathbb{C}, 1_{\mathbb{C}})$, then $\check{U}(E) = U(E) \times \{\pm 1\}$. Consequently, every irreducible Casselman-Wallach representation of $\text{Sp}(p, q)$ or $\text{Sp}_{2n}(\mathbb{C})$ is self-cotragredient.

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2. THE CLASSIFICATION OF QUATERNIONIC (ϵ, τ) -HERMITIAN A -MODULES

We fix a \mathbb{R} -algebra embedding $\mathbb{C} \hookrightarrow \mathbb{H}$ and thus view \mathbb{C} as a subalgebra of \mathbb{H} . Note that such an embedding is unique up to conjugations by \mathbb{H}^\times , and that complex conjugation is consistent with the quaternion conjugation under the embedding. Write $\mathbf{i} = \sqrt{-1} \in \mathbb{C}$, and fix an element $\mathbf{j} \in \mathbb{H}$ such that $\mathbf{j}^2 = -1$ and $\mathbf{j}x\mathbf{j}^{-1} = \bar{x}$ for all $x \in \mathbb{C}$. Denote by $\text{tr}_{\mathbb{H}/\mathbb{R}} : \mathbb{H} \rightarrow \mathbb{R}$ the reduced trace of \mathbb{H} .

Let (A, τ) be a commutative involutive algebra as in the introduction. In general, if $(E_i, \langle \cdot, \cdot \rangle_{E_i})$ is an (ϵ_i, τ) -hermitian A -module ($\epsilon_i = \pm 1$, $i = 1, 2$), for every $\varphi \in \text{Hom}_A(E_1, E_2)$, we define $\varphi^\tau \in \text{Hom}_A(E_2, E_1)$ by requiring that

$$(4) \quad \langle \varphi u, v \rangle_{E_2} = \langle u, \varphi^\tau v \rangle_{E_1}, \quad u \in E_1, v \in E_2.$$

Note that $(\varphi^\tau)^\tau = \epsilon_1 \epsilon_2 \varphi$.

Let $(E, \langle \cdot, \cdot \rangle_E)$ be a quaternionic (ϵ, τ) -hermitian A -module as in the introduction.

2.1. The case of $(A, \tau) = (\mathbb{R}, 1_{\mathbb{R}})$. Assume that $(A, \tau) = (\mathbb{R}, 1_{\mathbb{R}})$. Define a \mathbb{R} -bilinear map

$$\langle \cdot, \cdot \rangle_{E, \mathbb{H}} : E \times E \rightarrow \mathbb{H}$$

by requiring that

$$\text{tr}_{\mathbb{H}/\mathbb{R}}(\langle u, v \rangle_{E, \mathbb{H}} h) = \langle u, v h \rangle_E, \quad h \in \mathbb{H}, u, v \in E.$$

Then $\langle \cdot, \cdot \rangle_{E, \mathbb{H}}$ is a (non-degenerate) (ϵ, τ) -hermitian form on E , namely, it satisfies that

$$\langle u, v h \rangle_{E, \mathbb{H}} = \langle u, v \rangle_{E, \mathbb{H}} h, \quad \langle u, v \rangle_{E, \mathbb{H}} = \overline{\langle v, u \rangle_{E, \mathbb{H}}}, \quad h \in \mathbb{H}, u, v \in E.$$

On the other hand, $\langle \cdot, \cdot \rangle_{E, \mathbb{H}}$ determines $\langle \cdot, \cdot \rangle_E$ by the formula

$$\langle u, v \rangle_E = \text{tr}_{\mathbb{H}/\mathbb{R}}(\langle u, v \rangle_{E, \mathbb{H}}), \quad u, v \in \mathbb{H}.$$

Therefore, there is a one-one correspondence between the set of quaternionic (ϵ, τ) -hermitian forms, and the set of $(\epsilon, \bar{\cdot})$ -hermitian forms, on E . From the well known classification of $(\epsilon, \bar{\cdot})$ -hermitian forms, we conclude that

Proposition 2.1. *Assume that $(A, \tau) = (\mathbb{R}, 1_{\mathbb{R}})$. If $\epsilon = 1$, then every quaternionic (ϵ, τ) -hermitian A -module is isomorphic to a unique $(\mathbb{H}^{p+q}, \langle \cdot, \cdot \rangle_{\mathbb{H}^{p,q}})$, where $p, q \geq 0$, and*

$$\langle (u_1, u_2, \dots, u_{p+q}), (v_1, v_2, \dots, v_{p+q}) \rangle_{\mathbb{H}^{p,q}} = \text{tr}_{\mathbb{H}/\mathbb{R}} \left(\sum_{i \leq p} \bar{u}_i v_i - \sum_{j > p} \bar{u}_j v_j \right).$$

If $\epsilon = -1$, then every quaternionic (ϵ, τ) -hermitian A -module is isomorphic to a unique $(\mathbb{H}^n, \langle \cdot, \cdot \rangle_{\mathbb{H}^n, \mathbf{j}})$, where $n \geq 0$, and

$$\langle (u_1, u_2, \cdot, u_n), (v_1, v_2, \dots, v_n) \rangle_{\mathbb{H}^n, \mathbf{j}} = \text{tr}_{\mathbb{H}/\mathbb{R}} \left(\sum_{i=1}^n \bar{u}_i \mathbf{j} v_i \right).$$

2.2. The case of $(A, \tau) = (\mathbb{C}, 1_{\mathbb{C}})$. Assume that $(A, \tau) = (\mathbb{C}, 1_{\mathbb{C}})$. View \mathbb{H} as a \mathbb{C} - \mathbb{H} -bimodule. It is a quaternionic $(-1, 1_{\mathbb{C}})$ -hermitian \mathbb{C} -module under the form

$$(5) \quad \begin{aligned} \langle \cdot, \cdot \rangle_{\mathbb{H}, -1} : \mathbb{H} \times \mathbb{H} &\rightarrow \mathbb{C}, \\ (a + b\mathbf{j}, c + d\mathbf{j}) &\mapsto ad - bc, \quad a, b, c, d \in \mathbb{C}. \end{aligned}$$

Put

$$F := \text{Hom}_{\mathbb{C}, \mathbb{H}}(\mathbb{H}, E),$$

which is clearly a \mathbb{C} -vector space. It is a $(-\epsilon, 1_{\mathbb{C}})$ -hermitian \mathbb{C} -module under the form

$$\langle \varphi, \psi \rangle_F := \psi^\tau \circ \varphi \in \text{End}_{\mathbb{C}, \mathbb{H}}(\mathbb{H}) = \mathbb{C}.$$

On the other hand, $(E, \langle \cdot, \cdot \rangle_E)$ is determined by the $(-\epsilon, 1_{\mathbb{C}})$ -hermitian \mathbb{C} -module $(F, \langle \cdot, \cdot \rangle_F)$ since

$$E = F \otimes_{\mathbb{C}} \mathbb{H}$$

as a \mathbb{C} - \mathbb{H} -bimodule, and

$$\langle \cdot, \cdot \rangle_E = \langle \cdot, \cdot \rangle_F \langle \cdot, \cdot \rangle_{\mathbb{H}, -1}.$$

Therefore, there is a one-one correspondence between the set of isomorphism classes of quaternionic $(\epsilon, 1_{\mathbb{C}})$ -hermitian \mathbb{C} -module, and the set of isomorphism classes of $(-\epsilon, 1_{\mathbb{C}})$ -hermitian \mathbb{C} -module. By the classification of $(-\epsilon, 1_{\mathbb{C}})$ -hermitian \mathbb{C} -modules (that is, complex symplectic spaces or complex orthogonal spaces), we have

Proposition 2.2. *Assume that $(A, \tau) = (\mathbb{C}, 1)$. If $\epsilon = 1$, then every quaternionic (ϵ, τ) -hermitian A -module is isomorphic to a unique*

$$(\mathbb{C}^{2n} \otimes_{\mathbb{C}} \mathbb{H}, \langle \cdot, \cdot \rangle_{\mathbb{C}^{2n}, -1} \otimes \langle \cdot, \cdot \rangle_{\mathbb{H}, -1}),$$

where $n \geq 0$, and

$$\langle (u_1, u_2, \dots, u_{2n}), (v_1, v_2, \dots, v_{2n}) \rangle_{\mathbb{C}^{2n}, -1} = \sum_{i=1}^n (u_i v_{n+i} - u_{n+i} v_i).$$

If $\epsilon = -1$, then every quaternionic (ϵ, τ) -hermitian A -module is isomorphic to a unique

$$(\mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{H}, \langle, \rangle_{\mathbb{C}^n, 1} \otimes \langle, \rangle_{\mathbb{H}, -1}),$$

where $n \geq 0$, and

$$\langle (u_1, u_2, \dots, u_n), (v_1, v_2, \dots, v_n) \rangle_{\mathbb{C}^n, 1} = \sum_{i=1}^n u_i v_i.$$

2.3. The case of $(A, \tau) = (\mathbb{C}, \bar{\cdot})$. Assume that $(A, \tau) = (\mathbb{C}, \bar{\cdot})$. The argument is similar to that of the last section. View \mathbb{H} as a \mathbb{C} - \mathbb{H} -bimodule as before. It is a quaternionic $(1, \bar{\cdot})$ -hermitian \mathbb{C} -module under the form

$$\begin{aligned} \langle, \rangle_{\mathbb{H}, 1} : \mathbb{H} \times \mathbb{H} &\rightarrow \mathbb{C}, \\ (a + b\mathbf{j}, c + d\mathbf{j}) &\mapsto a\bar{c} + b\bar{d}, \quad a, b, c, d \in \mathbb{C}. \end{aligned}$$

The space $F := \text{Hom}_{\mathbb{C}, \mathbb{H}}(\mathbb{H}, E)$ is an $(\epsilon, \bar{\cdot})$ -hermitian \mathbb{C} -module under the form

$$\langle \varphi, \psi \rangle_F := \psi^\tau \circ \varphi \in \text{End}_{\mathbb{C}, \mathbb{H}}(\mathbb{H}) = \mathbb{C}.$$

On the other hand, (E, \langle, \rangle_E) is determined by (F, \langle, \rangle_F) since

$$E = F \otimes_{\mathbb{C}} \mathbb{H}$$

as a \mathbb{C} - \mathbb{H} -bimodule, and

$$\langle, \rangle_E = \langle, \rangle_F \langle, \rangle_{\mathbb{H}, 1}.$$

Therefore, by the classification of $(\epsilon, \bar{\cdot})$ -hermitian \mathbb{C} -modules (that is, the usual hermitian or skew hermitian spaces), we have

Proposition 2.3. *Assume that $(A, \tau) = (\mathbb{C}, \bar{\cdot})$. If $\epsilon = 1$, then every quaternionic (ϵ, τ) -hermitian A -module is isomorphic to a unique*

$$(\mathbb{C}^{p+q} \otimes_{\mathbb{C}} \mathbb{H}, \langle, \rangle_{\mathbb{C}^{p,q}, 1} \otimes \langle, \rangle_{\mathbb{H}, 1}),$$

where $p, q \geq 0$, and

$$\langle (u_1, u_2, \dots, u_{p+q}), (v_1, v_2, \dots, v_{p+q}) \rangle_{\mathbb{C}^{p,q}, 1} = \sum_{i \leq p} u_i \bar{v}_i - \sum_{j > p} u_j \bar{v}_j.$$

If $\epsilon = -1$, then every quaternionic (ϵ, τ) -hermitian A -module is isomorphic to a unique

$$(\mathbb{C}^{p+q} \otimes_{\mathbb{C}} \mathbb{H}, \langle, \rangle_{\mathbb{C}^{p,q}, -1} \otimes \langle, \rangle_{\mathbb{H}, 1}),$$

where $p, q \geq 0$, and

$$\langle (u_1, u_2, \dots, u_{p+q}), (v_1, v_2, \dots, v_{p+q}) \rangle_{\mathbb{C}^{p,q}, -1} = \sum_{i \leq p} \mathbf{i} u_i \bar{v}_i - \sum_{j > p} \mathbf{i} u_j \bar{v}_j.$$

2.4. The case of $(A, \tau) = (\mathbb{R} \times \mathbb{R}, \tau_{\mathbb{R}})$. Assume that $(A, \tau) = (\mathbb{R} \times \mathbb{R}, \tau_{\mathbb{R}})$. Write $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Then we have a decomposition

$$E = E_1 \oplus E_2$$

of E as a right \mathbb{H} -vector space, where $E_1 = e_1 E$ and $E_2 = e_2 E$. It is easy to see that there is a unique non-degenerate \mathbb{R} -bilinear map

$$\langle \cdot, \cdot \rangle_{E_1, E_2} : E_1 \times E_2 \rightarrow \mathbb{R}$$

satisfying

$$\langle uh, v \rangle_{E_1, E_2} = \langle u, v\bar{h} \rangle_{E_1, E_2}, \quad u \in E_1, v \in E_2, h \in \mathbb{H}$$

such that

$$\langle u_1 + u_2, v_1 + v_2 \rangle_E = (\langle u_1, v_2 \rangle_{E_1, E_2}, \epsilon \langle v_1, u_2 \rangle_{E_1, E_2}), \quad u_1, v_1 \in E_1, u_2, v_2 \in E_2.$$

Note that the right \mathbb{H} -vector space E_2 is determined by the right \mathbb{H} -vector space E_1 together with the pairing $\langle \cdot, \cdot \rangle_{E_1, E_2}$. Namely, $E_2 = \text{Hom}_{\mathbb{R}}(E_1, \mathbb{R})$ as right \mathbb{H} -vector spaces. Therefore, we have

Proposition 2.4. *Assume that $(A, \tau) = (\mathbb{R} \times \mathbb{R}, \tau_{\mathbb{R}})$. Then every quaternionic (ϵ, τ) -hermitian A -module is isomorphic to a unique*

$$(\mathbb{H}^n \oplus \mathbb{H}^n, \langle \cdot, \cdot \rangle_{\mathbb{H}^{2n}, \mathbb{R}, \epsilon}),$$

where $n \geq 0$, and

$$\langle (u_1, u_2, \dots, u_{2n}), (v_1, v_2, \dots, v_{2n}) \rangle_{\mathbb{H}^{2n}, \mathbb{R}, \epsilon} = (\text{tr}_{\mathbb{H}/\mathbb{R}}(\sum_{i=1}^n \bar{u}_i v_{n+i}), \epsilon \text{tr}_{\mathbb{H}/\mathbb{R}}(\sum_{i=1}^n \bar{v}_i u_{n+i})).$$

2.5. The case of $(A, \tau) = (\mathbb{C} \times \mathbb{C}, \tau_{\mathbb{C}})$. Note that every finitely generated \mathbb{C} - \mathbb{H} -bimodule is isomorphic to some \mathbb{H}^n . Now the argument is the same as in last section. We skip the details and record the result:

Proposition 2.5. *Assume that $(A, \tau) = (\mathbb{C} \times \mathbb{C}, \tau_{\mathbb{C}})$. Then every quaternionic (ϵ, τ) -hermitian A -module is isomorphic to a unique*

$$(\mathbb{H}^n \oplus \mathbb{H}^n, \langle \cdot, \cdot \rangle_{\mathbb{H}^{2n}, \mathbb{C}, \epsilon}),$$

where $n \geq 0$, and

$$\langle (u_1, u_2, \dots, u_{2n}), (v_1, v_2, \dots, v_{2n}) \rangle_{\mathbb{H}^{2n}, \mathbb{C}, \epsilon} = (\sum_{i=1}^n \langle u_i, v_{n+i} \rangle_{\mathbb{H}, -1}, \epsilon \sum_{i=1}^n \langle v_i, u_{n+i} \rangle_{\mathbb{H}, -1}).$$

Here the complex symplectic form $\langle \cdot, \cdot \rangle_{\mathbb{H}, -1} : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{C}$ is the one in (5).

We get Proposition 1.1 by combining Propositions 2.1-2.5.

3. QUATERNIONIC (ϵ, τ) -HERMITIAN (\mathfrak{sl}_2, A) -MODULES

By an (ϵ, τ) -hermitian (\mathfrak{sl}_2, A) -module, we mean an (ϵ, τ) -hermitian A -module E , together with a Lie algebra action

$$(6) \quad \mathfrak{sl}_2(\mathbb{R}) \times E \rightarrow E, \quad x, v \mapsto xv,$$

which is \mathbb{R} -linear on the first factor, A -linear on the second factor, and satisfies

$$\langle xu, v \rangle_E + \langle u, xv \rangle_E = 0, \quad x \in \mathfrak{sl}_2(\mathbb{R}), u, v \in E.$$

If E is furthermore a quaternionic (ϵ, τ) -hermitian A -module, and the action (6) is \mathbb{H} -linear on the second factor as well, we say that E is a quaternionic (ϵ, τ) -hermitian (\mathfrak{sl}_2, A) -module.

For every positive integer d , fix an irreducible real $\mathfrak{sl}_2(\mathbb{R})$ -module F_d of dimension d , which is unique up to isomorphism. We also fix a non-zero $\mathfrak{sl}_2(\mathbb{R})$ -invariant bilinear map

$$\langle \cdot, \cdot \rangle_{F_d} : F_d \times F_d \rightarrow \mathbb{R},$$

which is unique up to scalar multiplication. The form $\langle \cdot, \cdot \rangle_{F_d}$ is non-degenerate. It is symmetric if d odd, and is skew symmetric if d is even. The space

$$A \otimes_{\mathbb{R}} F_d$$

is obviously an $((-1)^{d-1}, \tau)$ -hermitian (\mathfrak{sl}_2, A) -module with the form

$$\langle a \otimes u, b \otimes v \rangle_{A \otimes_{\mathbb{R}} F_d} := ab^{\tau} \langle u, v \rangle_{F_d}$$

Now let E be a quaternionic (ϵ, τ) -hermitian (\mathfrak{sl}_2, A) -module. Denote by $E(d)$ the sum of irreducible $\mathfrak{sl}_2(\mathbb{R})$ -submodules of E which are isomorphic to F_d . It is again a quaternionic (ϵ, τ) -hermitian (\mathfrak{sl}_2, A) -module, and we have an orthogonal decomposition

$$E = \bigoplus_{d \geq 1} E(d).$$

The space

$$E(d)^{\circ} := \text{Hom}_{\mathfrak{sl}_2(\mathbb{R})}(F_d, E(d)) = \text{Hom}_{\mathfrak{sl}_2(\mathbb{R}), A}(A \otimes_{\mathbb{R}} F_d, E(d))$$

is checked to be a quaternionic $((-1)^{d-1}\epsilon, \tau)$ -hermitian A -module under the form

$$\langle \varphi, \psi \rangle_{E(d)^{\circ}} := \psi^{\tau} \circ \varphi \in \text{End}_{\mathfrak{sl}_2(\mathbb{R}), A}(A \otimes_{\mathbb{R}} F_d) = A.$$

On the other hand, the quaternionic (ϵ, τ) -hermitian (\mathfrak{sl}_2, A) -module $E(d)$ is determined by the quaternionic $((-1)^{d-1}\epsilon, \tau)$ -hermitian A -module $E(d)^{\circ}$ since

$$E(d) = E(d)^{\circ} \otimes_{\mathbb{R}} F_d$$

as a $\mathfrak{sl}_2(\mathbb{R})$ - A - \mathbb{H} -module, and

$$\langle \cdot, \cdot \rangle_E|_{E(d) \times E(d)} = \langle \cdot, \cdot \rangle_{E(d)^\circ} \otimes \langle \cdot, \cdot \rangle_{F_d}.$$

In conclusion, we have

$$(7) \quad E = \bigoplus_{d \geq 1} E(d)^\circ \otimes_{\mathbb{R}} F_d$$

as a quaternionic (ϵ, τ) -hermitian (\mathfrak{sl}_2, A) -module.

4. PROOF OF THEOREM 1.2

Let E be an (ϵ, τ) -hermitian A -module, or a quaternionic (ϵ, τ) -hermitian A -module, or an (ϵ, τ) -hermitian (\mathfrak{sl}_2, A) -module, or a quaternionic (ϵ, τ) -hermitian (\mathfrak{sl}_2, A) -module. We define E_τ , which is a space with the same kinds of structures as that of E , as follows. In any case, $E_\tau = E$ as a real vector space, and when E is quaternionic, $E_\tau = E$ as a right \mathbb{H} -vector space. For every $v \in E$, write $v_\tau := v$, viewed as a vector in E_τ . Then the scalar multiplication on E_τ is

$$av_\tau := (a^\tau v)_\tau, \quad a \in A, v \in E,$$

and the hermitian form is

$$\langle u_\tau, v_\tau \rangle_{E_\tau} := \langle v, u \rangle_E, \quad u, v \in E.$$

When E is equipped with a $\mathfrak{sl}_2(\mathbb{R})$ -action, the $\mathfrak{sl}_2(\mathbb{R})$ -action on E_τ is given by

$$\mathbf{h}v_\tau := (\mathbf{h}v)_\tau, \quad \mathbf{e}v_\tau := -(\mathbf{e}v)_\tau, \quad \mathbf{f}v_\tau := -(\mathbf{f}v)_\tau, \quad v \in E,$$

where

$$\mathbf{h} := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{e} := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{f} := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

which form a basis of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$.

Lemma 4.1. *If E be an (ϵ, τ) -hermitian A -module, or a quaternionic (ϵ, τ) -hermitian A -module, or an (ϵ, τ) -hermitian (\mathfrak{sl}_2, A) -module, or a quaternionic (ϵ, τ) -hermitian (\mathfrak{sl}_2, A) -module. Then, E_τ is isomorphic to E .*

Proof. The lemma is proved in [Sun2, Proposition 1.2 and 1.5] when E is an (ϵ, τ) -hermitian A -module, or a (ϵ, τ) -hermitian (\mathfrak{sl}_2, A) -module.

Without lose of generality, assume that (A, τ) is simple. The lemma follows easily from Proposition 1.1 if E is a quaternionic (ϵ, τ) -hermitian A -module.

Now assume that E is a quaternionic (ϵ, τ) -hermitian (\mathfrak{sl}_2, A) -module. Write

$$E = \bigoplus_{d \geq 1} E(d)^\circ \otimes_{\mathbb{R}} F_d$$

as in (7). Recall that $E(d)^\circ$ is a quaternionic $((-1)^{d-1}\epsilon, \tau)$ -hermitian A -module, and F_d is a $((-1)^{d-1}, 1_{\mathbb{R}})$ -hermitian $(\mathfrak{sl}_2, \mathbb{R})$ -module. Therefore

$$\begin{aligned}
E_\tau &\cong \left(\bigoplus_{d \geq 1} E(d)^\circ \otimes_{\mathbb{R}} F_d \right)_\tau \\
&\cong \bigoplus_{d \geq 1} (E(d)^\circ)_\tau \otimes_{\mathbb{R}} (F_d)_\tau \\
&\cong \bigoplus_{d \geq 1} E(d)^\circ \otimes_{\mathbb{R}} F_d \\
&= E.
\end{aligned}$$

□

Now we are ready to prove Theorem 1.2. Let E be a quaternionic (ϵ, τ) -hermitian A -module. Note that the isomorphism $E \cong E_\tau$ amounts to saying that there is an element of the form $(g, -1)$ in $\check{U}(E)$, that is, $U(E)$ has index two in $\check{U}(E)$. This proves the first statement of Theorem 1.2.

Let $x \in U(E)$. Write $x = su$ for the Jordan decomposition of x , where $s \in U(E)$ is semisimple, and $u \in U(E)$ is unipotent. Denote by A_s the subalgebra of $\text{End}_{A, \mathbb{H}}(E)$ generated by s , s^{-1} and scalar multiplications by A . Recall that as a special case of (4), we have defined an involution τ on $\text{End}_A(E)$. The algebra A_s is τ -stable, and the pair (A_s, τ) is a commutative involutive algebra. Write $E_s := E$, viewed as an A_s - \mathbb{H} -bimodule, equipped with the \mathbb{R} -bilinear map

$$\langle \cdot, \cdot \rangle_{E_s} : E_s \times E_s \rightarrow A_s$$

such that

$$\text{tr}_{A_s/\mathbb{R}}(a \langle u, v \rangle_{E_s}) = \text{tr}_{A/\mathbb{R}}(\langle au, v \rangle_E), \quad u, v \in E, \quad a \in A_s.$$

Then E_s becomes a quaternion (ϵ, τ) -hermitian A_s -modules, and we have that

$$\check{U}(E_s) = \{(g, \delta) \in \check{U}(E) \mid gsg^{-1} = s^\delta\}.$$

Use Jacobson-Morozov Theorem, we choose an action of $\mathfrak{sl}_2(\mathbb{R})$ on E_s such that it makes E_s a quaternionic (ϵ, τ) -hermitian (\mathfrak{sl}_2, A_s) -module, and that the exponential of the action of \mathfrak{e} coincides with u . Since $(E_s)_\tau$ is isomorphic to E_s as a quaternionic (ϵ, τ) -hermitian (\mathfrak{sl}_2, A_s) -module, there is an element $\check{g} = (g, -1) \in \check{U}(E_s)$ such that $gug^{-1} = u^{-1}$. This proves Theorem 1.2 since $\check{g} \in \check{U}(E) \setminus U(E)$ and $\check{g}x\check{g}^{-1} = x^{-1}$.

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